

# STABILITY OF MULTIPLICATION OPERATORS AND MULTIPLICATION SEMIGROUPS

FATIH BAYAZIT, RETHA HEYMANN

**ABSTRACT.** We investigate uniform, strong and almost weak stability properties of multiplication semigroups on Banach space valued  $L^p$ -spaces. We show that, under certain conditions, these properties can be characterized by analogous properties of the pointwise semigroups.

In this paper we investigate stability properties of multiplication semigroups consisting of multiplication operators on spaces of vector valued functions. Our aim is to understand how stability properties of such semigroups are related to those of the corresponding pointwise semigroups, as explained below. We also include a section in which we state analogous results for multiplication operators.

Multiplication semigroups (or operators) on spaces of Banach space valued function spaces have been studied by, e.g., Graser [7], Holderrieth [9], Hardt and Wagenführer [8], and Arendt and Thomaschewski [1]. See also [10, Section 4] and [13]. Qualitative properties of such semigroups, e.g. various stability concepts, are of great interest in control theory (e.g. [3]). In fact, motivated by such applications, Hans Zwart has proved a characterization of strong stability of a multiplication semigroup in the finite dimensional case (see [14]), while so-called polynomial stability of multiplication semigroups is characterized in Theorem 4.4 of [2].

Throughout this text we assume that the measure space  $(\Omega, \Sigma, \mu)$  is  $\sigma$ -finite and has the finite subset property, i.e., for every  $Y \in \Sigma$  with  $\mu(Y) > 0$  there exists a  $Z \in \Sigma$  such that  $Z \subseteq Y$  and  $0 < \mu(Z) < \infty$ . Moreover, if  $X$  is a Banach space,  $L^p(\Omega, X)$  denotes the Bochner space  $L^p(\Omega, \Sigma, \mu; X)$  for each  $1 \leq p \leq \infty$ .

**Definition 1** (Multiplication Operator, Pointwise Operator). [12] Let  $X$  be a separable Banach space and let  $M : \Omega \rightarrow \mathcal{L}(X)$  such that, for every  $x \in X$ , the function

$$\Omega \ni s \mapsto M(s)x \in X$$

is Bochner measurable. The *multiplication operator*  $\mathcal{M}$  on  $L^p(\Omega, X)$ , with  $1 \leq p \leq \infty$ , is defined by

$$(\mathcal{M}f)(s) := M(s)f(s) \text{ for all } s \in \Omega$$

with

$$D(\mathcal{M}) = \{f \in L^p(\Omega, X) \mid M(\cdot)f(\cdot) \in L^p(\Omega, X)\}.$$

In this context, we call the operators  $M(s)$  with  $s \in \Omega$  the *pointwise operators* on  $X$ .

---

*Key words and phrases.* multiplication operators, operator multipliers, multiplication semigroups, stability (uniform, strong, almost weak).

The authors would like to express their sincere gratitude to Rainer Nagel for his ideas, encouragement, support and interest in our project.

*Remark 2.* The Bochner measurability of  $s \mapsto M(s)x$  for all  $x \in X$  implies that the function  $M(\cdot)f(\cdot)$  is also measurable if  $f \in L^p(\Omega, X)$ , see [12, Lemma 2.2.9].

**Definition 3** (Multiplication Semigroup). If a  $C_0$ -semigroup  $(\mathcal{M}(t))_{t \geq 0}$  on  $L^p(\Omega, X)$  consists of multiplication operators on  $L^p(\Omega, X)$ , it is called a *multiplication semigroup*. (See [6], for instance, for the general theory of  $C_0$ -semigroups.)

*Remark 4.* If  $(\mathcal{A}, D(\mathcal{A}))$  is the generator of the multiplication semigroup  $(\mathcal{M}(t))_{t \geq 0}$ , then there exists a family of operators  $A(s)$  with domain  $D(A(s))$ , on  $X$  such that  $A(s)$  is the generator of a  $C_0$ -semigroup on  $X$  for all  $s \in \Omega \setminus N$  for some null set  $N$  (see [12, Theorem 2.3.15, p. 49]). We call these semigroups on  $X$  the *pointwise semigroups*.

We denote the multiplication semigroup  $(\mathcal{M}(t))_{t \geq 0}$  by  $(e^{t\mathcal{A}})_{t \geq 0}$  and the pointwise semigroups by  $(e^{tA(s)})_{t \geq 0}$  for all  $s \in \Omega \setminus N$ .

Furthermore, for every  $t \geq 0$ , the function from  $\Omega$  to  $\mathcal{L}(X)$ ,  $s \mapsto e^{tA(s)}$ , is measurable and the operator  $e^{t\mathcal{A}}$  is the corresponding multiplication operator on  $L^p(\Omega, X)$  see [12, 2.3.12].

Before we can determine stability properties, we need to know when a multiplication operator is bounded and what its norm is. This has been determined by Thomaschewski (see [12]), but in the following lemma we follow the reasoning of Klaus-J. Engel in his proof for the case where  $X = \mathbb{C}^n$  (see [5, Chapter IX, Proposition 1.3]).

**Lemma 5.** [12, Proposition 2.2.14, p. 35]

Let  $\mathcal{M}$  be a multiplication operator on  $L^p(\Omega, X)$  induced by  $M$  as in Definition 1. The operator  $\mathcal{M}$  is bounded if and only if the function  $M(\cdot)$  is essentially bounded. If this is the case, we have that

$$\begin{aligned} \|\mathcal{M}\| &= \operatorname{ess\,sup}_{s \in \Omega} \|M(s)\| \\ &:= \inf \{C \geq 0 \mid \mu(\{s \in \Omega \mid \|M(s)\| > C\}) = 0\}. \end{aligned}$$

*Proof.* Assume that, for some constant  $K \geq 0$ , there exists a set  $N \subseteq \Omega$  such that  $\mu(N) > 0$  and  $\|M(s)\| > K$  for all  $s \in N$ . Because of the finite subset property, we can assume, without loss of generality, that  $\mu(N) < \infty$ .

Let  $(x_n)$  be a dense sequence in the unit sphere of the separable Banach space  $X$ . For every  $s \in N$  we can find an  $x_n$  such that  $\|M(s)x_n\| > K$ . Therefore, if we define  $N_n := \{s \in \Omega \mid \|M(s)x_n\| > K\}$  for every  $n \in \mathbb{N}$ , we have that

$$N = \bigcup_{n \in \mathbb{N}} N_n.$$

Because  $\mu(N) > 0$  and  $N$  is the countable union of the sets  $N_n$ , there is at least one element of  $N$ , say  $k$ , such that

$$0 < \mu(N_k) = \mu(\{s \in \Omega \mid \|M(s)x_k\| > K\}) < \infty.$$

We now define the function  $f_x \in L^p(\Omega, X)$  by

$$f_x(s) := \mathbb{1}_{N_k}(s)x_k, \quad s \in \Omega,$$

where  $\mathbb{1}_{N_k} : \Omega \rightarrow \{0, 1\}$  is the indicator function of the set  $N_k$ . Then  $\|f_x\|^p = \int_{\Omega} \|f_x(s)\|^p d\mu(s) = \int_{N_k} 1^p d\mu(s) = \mu(N_k)$  and hence

$$\begin{aligned} \|\mathcal{M}f_x\|^p &= \int_{\Omega} \|M(s)f_x(s)\|^p d\mu(s) \\ &= \int_{N_k} \|M(s)x_k\|^p d\mu(s) \\ &> \left(K\mu(N_k)^{\frac{1}{p}}\right)^p \\ &= (K\|f_x\|)^p, \end{aligned}$$

so that  $\|\mathcal{M}\| > K$ .

From this it immediately follows that

$$\operatorname{ess\,sup}_{s \in \Omega} \|M(s)\| \leq \|\mathcal{M}\|$$

if  $\mathcal{M}$  is bounded.

It is easy to see that, if  $\operatorname{ess\,sup}_{s \in \Omega} \|M(s)\| < K$ , then  $\|\mathcal{M}\| < K$ . We conclude that  $\mathcal{M}$  is bounded if and only if  $M(\cdot)$  is essentially bounded and in this case  $\|\mathcal{M}\| = \operatorname{ess\,sup}_{s \in \Omega} \|M(s)\|$ .  $\square$

*Remark 6.* Although  $\mathcal{M}$  is regarded as an operator on the space  $L^p(\Omega, X)$  for some fixed  $1 \leq p \leq \infty$ , its norm  $\|\mathcal{M}\|$  is independent of  $p$ .

We are interested in the extent to which stability properties of the pointwise semigroups determine stability properties (see the sections below) of the corresponding multiplication semigroup. This is not always the case, as we see in the following example which is a modification of Zabzyk's classical counterexample to the spectral mapping theorem for  $C_0$ -semigroups (See [6] p. 273, Counterexample 3.4).

*Example 7.* Let  $H$  be the Hilbert space  $\ell^2(\mathbb{N})$ . For each  $n \in \mathbb{N}$  we consider the operator  $A_n$  on  $H$  with  $n$ -dimensional range defined by the infinite matrix

$$A_n = \begin{pmatrix} in - \frac{1}{n} & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & 0 & 0 & 0 & \dots \\ \vdots & & \ddots & 1 & 0 & 0 & \dots \\ 0 & \dots & 0 & in - \frac{1}{n} & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the generated semigroup  $(e^{tA_n})_{t \geq 0}$ . Then we have that the spectral bound  $s(A_n) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_n)\}$  is equal to  $-\frac{1}{n}$  and therefore each semigroup  $(e^{tA_n})_{t \geq 0}$  is uniformly stable. We now naturally consider the matrix  $A_n$  and the semigroup  $(e^{tA_n})_{t \geq 0}$  as elements belonging to  $\mathcal{L}(\ell^2(\mathbb{N}))$ . Then,  $(e^{tA_n})_{t \geq 0}$  is uniformly stable on  $\ell^2(\mathbb{N})$  as well.

The corresponding multiplication semigroup  $(e^{tA})_{t \geq 0}$  on  $\ell^2(\mathbb{N}, H)$  is induced by

the map  $\mathbb{N} \ni n \mapsto e^{tA_n} \in \mathcal{L}(H)$ . Since  $in - \frac{1}{n} \in P\sigma(\mathcal{A})$  for all  $n \in \mathbb{N}$ , we obtain

$$s(\mathcal{A}) \geq 0.$$

Hence,  $(e^{tA})_{t \geq 0}$  is not uniformly stable. In fact,  $(e^{tA})_{t \geq 0}$  is not even bounded.

More examples are discussed in [3, Section 2].

## 1. UNIFORM STABILITY

This short section is devoted to the strongest notion of stability, namely uniform stability.

**Definition 8.** [6, Definition V.1.1, p. 296] A  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  of operators on a Banach space is called *uniformly stable* if  $\|T(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

Lemma 5 immediately leads to the following characterization of uniform stability for multiplication semigroups.

**Theorem 9.** *Let  $(e^{tA})_{t \geq 0}$  be a multiplication semigroup on  $L^p(\Omega, X)$ . Then the following are equivalent.*

- (a) *The semigroup  $(e^{tA})_{t \geq 0}$  is uniformly stable.*
- (b) *The pointwise semigroups converge to 0 in norm, uniformly in  $s$ , i.e.,  $\operatorname{ess\,sup}_{s \in \Omega} \|e^{tA(s)}\| \rightarrow 0$  as  $t \rightarrow \infty$ .*
- (c) *At some  $t_0 > 0$ , the spectral radii  $r(e^{t_0 A(s)})$  of the pointwise semigroups satisfy  $\operatorname{ess\,sup}_{s \in \Omega} r(e^{t_0 A(s)}) < 1$ .*
- (d) *There exist constants  $M > 1$  and  $\epsilon > 0$  such that  $\|e^{tA(s)}\| \leq Me^{-t\epsilon}$  for all  $t \geq 0$  and almost all  $s$ .*

*Remark 10.* Note that the uniform stability of a multiplication semigroup is independent of the value of  $p$ .

## 2. STRONG STABILITY

A weaker notion than uniform stability of  $C_0$ -semigroups is strong stability.

**Definition 11.** [6, Definition V.1.1, p. 296] A  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  of operators on a Banach space is called *strongly stable* if  $\|T(t)x\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in X$ .

We now show that a multiplication semigroup is strongly stable if and only if the pointwise semigroups are uniformly bounded in  $s$  and strongly stable almost everywhere. The backward implication was proved by Curtain-Iftime-Zwart in [3] for the special case where  $\Omega = [0, 1]$ ,  $p = 2$  and  $X = \mathbb{C}^n$ . The other implication was conjectured in the same paper and has since been proved by Hans Zwart (see Theorem on p. 3 in [14]).

**Theorem 12.** *Suppose that the multiplication operator  $\mathcal{A}$  generates a  $C_0$ -semigroup of multiplication operators  $(e^{tA})_{t \geq 0}$  on  $L^p(\Omega, X)$  such that  $\|e^{tA}\| \leq M$  for all  $t \geq 0$  and some constant  $M > 0$ . Then the following are equivalent.*

- (a) *The semigroup  $(e^{tA})_{t \geq 0}$  is strongly stable.*
- (b) *The pointwise semigroups  $(e^{tA(s)})_{t \geq 0}$  on  $X$  are strongly stable for almost all  $s \in \Omega$ .*

*Proof.* Note that, by Lemma 5, we have  $\|e^{tA}\| \leq M$  for all  $t \geq 0$  if and only if  $\|e^{tA(s)}\| \leq M$  for almost all  $s \in \Omega$  and for all  $t \geq 0$ .

(a)  $\implies$  (b): Assume that  $\|e^{tA}f\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $f \in L^p(\Omega, X)$ .

Choose an arbitrary  $x \in X$  and define the function  $f_x : \Omega \rightarrow X$  by

$$f_x(s) := x$$

for all  $s \in \Omega$ . Since  $\Omega$  is  $\sigma$ -finite, we can write  $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$  where  $\mu(\Omega_n) < \infty$  for every  $n \in \mathbb{N}$ . Then  $f_x|_{\Omega_n} \in L^p(\Omega, X)$  for every  $n \in \mathbb{N}$ . By assumption, we have that

$$\|e^{tA}f_x|_{\Omega_n}\| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for every  $n \in \mathbb{N}$ . Therefore, the Riesz subsequence theorem (see, e.g., proof of [11, Chapter I, Theorem 3.12]) implies that, for every  $n \in \mathbb{N}$ , there exists a sequence  $(t_k) \in \mathbb{R}_+$  tending to infinity as  $k \rightarrow \infty$ , such that the functions  $e^{t_k A}f_x|_{\Omega_n}$  converge pointwise to 0, almost everywhere, i.e.

$$(13) \quad \|e^{t_k A(s)}f_x|_{\Omega_n}(s)\|_X = \|e^{t_k A(s)}x\|_X \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

for  $s \in \Omega_n \setminus N_{(x,n)}$ , where  $N_{(x,n)}$  is a subset of  $\Omega$  of measure 0. We now show that (13) implies that  $\|e^{tA(s)}x\|_X \rightarrow 0$  as  $t \rightarrow \infty$  for  $s \in \Omega_n \setminus N_{(x,n)}$ .

Let  $\epsilon > 0$ . Then  $\|e^{t_k A(s)}x\|_X < \frac{1}{M}\epsilon$  for all  $k$  greater than or equal to some  $k_\epsilon \in \mathbb{N}$ . For each  $t > t_{k_\epsilon}$ , we can write  $t = t_{k_\epsilon} + r$  where  $r \in \mathbb{R}_+$ . Then we have that

$$\begin{aligned} \|e^{tA(s)}x\|_X &= \|e^{(r+t_{k_\epsilon})A(s)}x\|_X \\ &= \left\| \left( e^{rA(s)} \right) \left( e^{t_{k_\epsilon}A(s)}x \right) \right\|_X \\ &\leq \|e^{rA(s)}\| \left\| e^{t_{k_\epsilon}A(s)}x \right\|_X \\ &\leq M \left\| e^{t_{k_\epsilon}A(s)}x \right\|_X \\ &\leq M \frac{1}{M}\epsilon \\ &= \epsilon. \end{aligned}$$

Hence,

$$(14) \quad \|e^{tA(s)}x\|_X \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all  $s \in \Omega_n \setminus N_{(x,n)}$ . It follows that (14) holds for all  $s \in \Omega \setminus (\cup_{n \in \mathbb{N}} N_{(x,n)})$ . Observe that  $\cup_{n \in \mathbb{N}} N_{(x,n)}$ , being a countable union of null sets, is also a null set and hence we have the convergence (14) almost everywhere.

For each  $x \in X$ , we can find a null set  $N_x = \cup_{n \in \mathbb{N}} N_{(x,n)}$  such that (14) holds for all  $s \in \Omega \setminus N_x$ . Now, choose any countable, dense subset  $C \subset X$ . Then (14) holds for each  $x \in C$ , for all  $s \in \Omega \setminus (\cup_{x \in C} N_x)$ . Note that  $\cup_{x \in C} N_x$  is also null set. So we have that (14) holds for all  $x$  in a dense subset of  $X$ , almost everywhere. Because the semigroups  $(e^{tA(s)})_{t \geq 0}$  are bounded, it follows that (14) holds for all  $x \in X$ , almost everywhere, which is what we wanted to prove.

(b)  $\implies$  (a): Assume that  $\|e^{tA(s)}x\|_X \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in X$  and almost all  $s \in \Omega$ .

Choose an arbitrary function  $f \in L^p(\Omega, X)$ . Then

$$\left\| e^{tA(s)} f(s) \right\|_X^p \leq M^p \|f(s)\|_X^p$$

for almost all  $s \in \Omega$ . Hence, the functions  $\|e^{tA} f(\cdot)\|_X^p$  are dominated by the integrable function  $\|Mf(\cdot)\|_X^p$ . Because of our assumption, we know that  $\|e^{tA(s)} f(s)\|_X^p \rightarrow 0$  as  $t \rightarrow \infty$  for almost all  $s \in \Omega$ . It now follows from Lebesgue's dominated convergence theorem that

$$\int_{\Omega} \left\| e^{tA(s)} f(s) \right\|_X^p ds = \|e^{tA} f\|^p \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Thus the proof is complete.  $\square$

*Remark 15.* Note that, as before, strong stability of a multiplication semigroup is independent of the value  $p$ , as long as  $1 \leq p < \infty$ .

### 3. ALMOST WEAK STABILITY

We now investigate the almost weak stability of the multiplication semigroup  $(e^{tA})_{t \geq 0}$  and the pointwise semigroups  $(e^{tA(s)})_{t \geq 0}$  with  $s \in \Omega$ . In order to define this stability concept, we mention that the density of a Lebesgue measurable subset of  $\mathbb{R}_+$  is  $d := \lim_{t \rightarrow \infty} \frac{\mu(M \cap [0, t])}{t}$  ( $\mu$  being the Lebesgue measure), whenever the limit exists.

**Definition 16** (Almost weak stability). Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup on a reflexive Banach space  $X$ . Then  $(T(t))_{t \geq 0}$  is called *almost weakly stable* if there exists a Lebesgue measurable set  $M \subset \mathbb{R}_+$  of density 1 such that

$$T(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad t \in M,$$

in the weak operator topology.

The main result of this section is a characterization of almost weak stability of a multiplication semigroup via the pointwise semigroups. Once again, since the stability is determined by the pointwise semigroups, the value of  $p$  is irrelevant as long as  $1 < p < \infty$ .

**Theorem 17.** Let the measure space  $(\Omega, \Sigma, \mu)$  be separable and let  $X$  be a reflexive, separable Banach space. Take  $(\mathcal{A}, D(\mathcal{A}))$  to be the generator of a bounded multiplication semigroup  $(e^{tA})_{t \geq 0}$  on  $L^p(\Omega, X)$  with  $1 < p < \infty$ , and let  $A(s)$  with  $s \in \Omega$  be the family of operators corresponding to  $\mathcal{A}$ .

If the pointwise semigroups  $(e^{tA(s)})_{t \geq 0}$  are almost weakly stable for almost all  $s \in \Omega$ , then the semigroup  $(e^{tA})_{t \geq 0}$  is almost weakly stable.

If the semigroup  $(e^{tA})_{t \geq 0}$  is almost weakly stable, then we have for each  $\lambda \in i\mathbb{R}$  that  $\lambda \in P\sigma(A(s))$  for all  $s$  in a zero set  $N_\lambda$ . Furthermore,  $\mu(\cup_{\lambda \in i\mathbb{R}} N_\lambda) = 0$  if and only if the pointwise semigroups  $(e^{tA(s)})_{t \geq 0}$  are almost weakly stable, almost everywhere.

In order to prove this theorem we need the following characterization of almost weak stability of a semigroup on a reflexive Banach space via the point spectrum of its generator.

**Lemma 18.** [4, Chapter II, Theorem 4.1] *Let  $(T(t))_{t \geq 0}$  be a bounded  $C_0$ -semigroup with generator  $(A, D(A))$  on a reflexive, separable Banach space  $X$ . Then the following are equivalent.*

- (1)  $(T(t))_{t \geq 0}$  is almost weakly stable.
- (2)  $P\sigma(A) \cap i\mathbb{R} = \emptyset$ , where  $P\sigma(A)$  is the point spectrum of  $A$ .

Moreover, we need some notions and results related to ergodic theory.

**Definition 19** (Cesàro mean; mean ergodic semigroup, mean ergodic projection). ([4, p. 20, Chapter I, Definition 2.18, 2.20]) Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup of operators on a Banach space  $X$  with generator  $(A, D(A))$ . For every  $t > 0$ , the Cesàro mean  $S(t) \in \mathcal{L}(X)$  is defined by

$$S(t)x := \frac{1}{t} \int_0^t T(s)x ds$$

for all  $x \in X$ . The semigroup  $(T(t))_{t \geq 0}$  is called *mean ergodic* if the Cesàro means converge pointwise as  $t$  tends to  $\infty$ . In this case the operator  $P \in \mathcal{L}(X)$  defined by

$$Px := \lim_{t \rightarrow \infty} S(t)x$$

is called the *mean ergodic projection* corresponding to  $(T(t))_{t \geq 0}$ .

*Remark 20.* [4, p. 21-22; Chapter I; Remark 2.21, Proposition 2.24 and Theorem 2.25] The mean ergodic projection  $P$  commutes with  $(T(t))_{t \geq 0}$  and is indeed a projection. A bounded  $(T(t))_{t \geq 0}$  is mean ergodic if and only if  $X = \ker A \oplus \overline{\text{ran } A}$ , where  $\ker A$  and  $\text{ran } A$  denote the kernel and range, respectively, of  $A$ . One also has that  $\text{ran } P = \ker A = \text{fix } (T(t))_{t \geq 0}$  and  $\ker P = \overline{\text{ran } A}$  where  $\text{fix } (T(t))_{t \geq 0}$  is the fixed space of  $(T(t))_{t \geq 0}$ .

We now characterize the eigenvalues on the imaginary axis  $i\mathbb{R}$  of the generator of a multiplication semigroup. The proof of Theorem 17 then follows easily, as indicated below.

**Proposition 21.** *Let the measure space  $(\Omega, \Sigma, \mu)$  be separable and let  $X$  be a reflexive, separable Banach space. Let  $(A, D(A))$  be the generator of a bounded multiplication semigroup  $(e^{tA})_{t \geq 0}$  on  $L^p(\Omega, X)$  with  $1 < p < \infty$ . Let  $A(s)$  with  $s \in \Omega$  be the family of operators corresponding to  $A$  and let  $\lambda \in i\mathbb{R}$ . Then,*

$$\lambda \in P\sigma(A) \iff \lambda \in P\sigma(A(s)) \text{ for } s \in Z,$$

where  $Z$  is a measurable subset of  $\Omega$  with  $\mu(Z) > 0$ .

*Proof.* It follows from von Neumann's Mean Ergodic Theorem (see [4, p. 22, Theorem 2.25] or [6, p. 340, Corollary V.4.6]) that we have convergence of the Cesàro means

$$\frac{1}{t} \int_0^t e^{\tau A(s)} x d\tau \rightarrow P(s)x \text{ as } t \rightarrow \infty$$

for all  $x \in X$  and almost all  $s \in \Omega$ , where each  $P(s)$  is the mean ergodic projection corresponding to  $(e^{tA(s)})_{t \geq 0}$  onto  $\ker A(s) = \text{fix } (e^{tA(s)})_{t \geq 0}$ . Since, for every  $x \in X$  and every  $t > 0$ , the function  $e^{tA(\cdot)}x$  is Bochner and hence weakly measurable,  $\frac{1}{t} \int_0^t e^{\tau A(\cdot)} x d\tau$  is also weakly measurable. By standard measure theory the pointwise limit of a sequence of weakly measurable functions is still weakly measurable.

Hence,  $P(\cdot)x$  is weakly measurable for every  $x \in X$  and therefore Bochner measurable. If a function  $f$  is in  $L^p(\Omega, X)$ , then, by Lebesgue's Dominated Convergence Theorem, the function  $P(\cdot)f(\cdot)$  is also in  $L^p(\Omega, X)$ , since  $\|P(s)\| = 1$  for almost all  $s \in \Omega$ . Let  $\mathcal{P}$  be the multiplication operator on  $L^p(\Omega, X)$  corresponding to the function  $P(\cdot)$ .

We now show that  $\mathcal{P}$  is the mean ergodic projection of  $(e^{t\mathcal{A}})_{t \geq 0}$  onto  $\text{fix}(e^{t\mathcal{A}})$ . If the function  $f$  is in  $\text{fix}(e^{t\mathcal{A}})$ , then  $e^{tA(s)}f(s) = f(s) = P(s)f(s)$  for all  $t \geq 0$  and almost all  $s \in \Omega$  and hence  $\mathcal{P}f = f$ . If the function  $f$  is in  $\overline{(\text{ran } \mathcal{A})}$ , then there exists a sequence of functions  $(g_n) \subset D(\mathcal{A})$  such that  $\mathcal{A}g_n \rightarrow f$  as  $n \rightarrow \infty$  in the  $L^p$  norm. Therefore, by the Riesz subsequence theorem,  $(g_n)$  has a subsequence  $(g_{n_k})$  such that  $(\mathcal{A}g_{n_k})(s) = A(s)g_{n_k}(s) \rightarrow f(s)$  for almost all  $s \in \Omega$ . That means that  $f(s) \in \overline{\text{ran } A(s)} = \ker P(s)$  for almost all  $s \in \Omega$  and hence  $f \in \ker \mathcal{P}$ .

Now, on the one hand, if  $0 \in P\sigma(A(s))$  for  $s \in Z$  with  $\mu(Z) > 0$ , then  $P(\cdot) \neq 0$ , and therefore  $\text{ess sup}_{s \in \Omega} \|P(s)\| > 0$ . By Lemma 5,  $\mathcal{P} \neq 0$  and therefore the kernel of  $\mathcal{A}$  is non-trivial. Hence  $0 \in P\sigma(\mathcal{A})$ .

On the other hand, if  $0 \notin P\sigma(A(s))$  for almost every  $s \in \Omega$ , then  $(\mathcal{A}f)(s) = A(s)f(s) = 0$  only if  $f(s) = 0$  almost everywhere.

By applying the above reasoning to the operator  $\lambda - \mathcal{A}$  for some  $\lambda \in i\mathbb{R}$ , we conclude that  $\lambda$  is an eigenvalue of  $A(s)$  for every  $s \in Z$ , where  $Z$  is a measurable set with  $\mu(Z) > 0$ , if and only if  $\lambda$  is an eigenvalue of  $\mathcal{A}$ .  $\square$

We are now ready to prove Theorem 17.

*Proof.* Using Lemma 18 almost weak stability of  $(e^{t\mathcal{A}})_{t \geq 0}$  is implied by  $P\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ , which, by Proposition 21, is equivalent to  $P\sigma(A(s)) \cap i\mathbb{R} = \emptyset$  almost everywhere. This in turn is equivalent to the fact that the pointwise semigroups  $(e^{tA(s)})_{t \geq 0}$  are almost weakly stable for almost all  $s \in \Omega$ .

The second part of the theorem also follows easily from Lemma 18 and Proposition 21.  $\square$

*Remark 22.* Again, this result is independent of  $1 < p < \infty$ .

We include a trivial example in the scalar case where the point spectrum  $P\sigma(A(s)) \neq \emptyset$  for each  $s \in \Omega$ , i.e. the none of the pointwise semigroups are almost weakly stable, whereas the pointsprum of  $\mathcal{A}$  is empty, which means that the semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is almost weakly stable.

*Example 23.* We use the notation as in Proposition 21 with  $\Omega = [0, 1]$ ,  $X = \mathbb{C}$  and  $A(s) := is$  for all  $s \in [0, 1]$ . Then  $is \in P\sigma(A(s))$  for all  $s \in [0, 1]$ , but  $P\sigma(\mathcal{A}) = \emptyset$ .

#### 4. STABILITY OF MULTIPLICATION OPERATORS

It is possible to develop analogous results for time-discrete semigroups of the form  $\{\mathcal{M}^n \mid n \in \mathbb{N}\}$  for a bounded multiplication operator  $\mathcal{M}$  on  $L^p(\Omega, X)$ .

The relevant stability properties are the following.

**Definition 24.** Let  $T$  be an operator on a Banach space  $X$ . Then  $T$  is

- (i) *uniformly stable* if  $\|T^n\| \rightarrow 0$  as  $n \rightarrow \infty$ .
- (ii) *strongly stable* if  $\|T^n f\| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $f \in X$ .



- (iii) *almost weakly stable* if  $\varphi(T^{n_k} f) \rightarrow 0$  as  $k \rightarrow \infty$  for all  $f \in X, \varphi \in X'$ , and sequences  $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$  of density 1, where  $X'$  denotes the continuous dual space of  $X$ .

Recall that the *density* of a sequence  $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$  is

$$d := \lim_{n \rightarrow \infty} \frac{|\{k : n_k < n\}|}{n},$$

if the limit exists and that a bounded linear operator  $T$  on a Banach space is called *power bounded* if  $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$ . By using methods analogous to those developed in Sections 1 – 3 we can characterize these stability properties of a power bounded multiplication operator  $\mathcal{M}$  through the pointwise operators  $M(\cdot)$ .

**Theorem 25.** *Let  $\mathcal{M}$  be a power-bounded multiplication operator on  $L^p(\Omega, X)$ .*

- (i) *Then  $\mathcal{M}$  is uniformly stable if and only if  $M(s)$  is uniformly stable for almost all  $s \in \Omega$  and  $\text{ess sup}_{s \in \Omega} \|M(s)^n\| \rightarrow 0$  as  $n \rightarrow \infty$ , or, equivalently, that  $\text{ess sup}_{s \in \Omega} r(M(s)) < 1$ .*
- (ii) *If  $\Omega$  is  $\sigma$ -finite, then  $\mathcal{M}$  is strongly stable if and only if  $M(s)$  is strongly stable for almost all  $s \in \Omega$ .*
- (iii) *Let the measure space  $(\Omega, \Sigma, \mu)$  be separable and let  $X$  be a reflexive, separable Banach space. Then  $\mathcal{M}$  is almost weakly stable if and only if  $M(s)$  is almost weakly stable for almost all  $s \in \Omega$ .*

## REFERENCES

- [1] W. ARENDT AND S. THOMASCHESKI, *Local operators and forms*, Positivity, 9 (2005), pp. 357–367.
- [2] A. BÁTKAI, K.-J. ENGEL, J. PRÜSS, AND R. SCHNAUBELT, *Polynomial stability of operator semigroups*, Math. Nachr., 279 (2006), pp. 1425–1440.
- [3] R. CURTAIN, O. V. IFTIME, AND H. ZWART, *System theoretic properties of a class of spatially invariant systems*, Automatica, 45 (2009), pp. 1619 – 1627.
- [4] T. EISNER, *Stability of Operators and Operator Semigroups*, vol. 209 of Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 2010.
- [5] K.-J. ENGEL, *Operator Matrices and Systems of Evolution Equations*. Manuscript, 1997.
- [6] K.-J. ENGEL AND R. NAGEL, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [7] T. GRASER, *Operator multipliers generating strongly continuous semigroups*, Semigroup Forum, 55 (1997), pp. 68–79.
- [8] V. HARDT AND E. WAGENFÜHRER, *Spectral properties of a multiplication operator*, Math. Nachr., 178 (1996), pp. 135–156.
- [9] A. HOLDERRIETH, *Matrix multiplication operators generating one parameter semigroups*, Semigroup Forum, 42 (1991), pp. 155–166.
- [10] D. MUGNOLO AND R. NITTKA, *Properties of representations of operators acting between spaces of vector-valued functions*, Positivity, 15 (2011), pp. 135–154.
- [11] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill Book Co., New York, second ed., 1974. McGraw-Hill Series in Higher Mathematics.
- [12] S. THOMASCHESKI, *Form methods for autonomous and nonautonomous Cauchy problems*, PhD-Thesis, Uni Ulm, 2003.
- [13] J. M. A. M. VAN NEERVEN, *Abstract multiplication semigroups*, Math. Z., 213 (1993), pp. 1–15.
- [14] H. ZWART, *Stability*, unpublished, (2008).

FATİH BAYAZIT, RETHA HEYMANN  
UNIVERSITÄT TÜBINGEN  
ARBEITSBEREICH FUNKTIONALANALYSIS  
MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT  
AUF DER MORGENSTELLE 10  
D-72076 TÜBINGEN

*E-mail addresses:*

faba@fa.uni-tuebingen.de  
rehe@fa.uni-tuebingen.de